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A symplectic context for level dynamics ☆

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Abstract

We consider a mathematical context which was suggested by quantum mechanical considerations of level dynamics. Although the situation is a general one, we restrict our attention to certain examples of physical relevance where explicit calculations are possible. Cases where M is the cotangent space of some Lie group or Lie algebra Q of operators on a finite-dimensional vector space are of particular interest.

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1. Introduction

1.1. Calculations with variables

Let us summarize the original physical discussion which was geared to describing the time evolution of a one dimension gas of particles with positions corresponding to eigenvalues of symmetric matrices [1–4]. This is in some sense a toy model, but is representative of an interesting general setting.

For this Q is the space of symmetric, $(n \times n)$-matrices with real entries and a pair $(X, Y) \in Q \times Q$ is regarded as a point X with a velocity Y. The evolution of the fictitious gas is as simple as possible: $(X, Y) \mapsto (X + tY, Y)$, which corresponds to $\dot{X} = Y$ and $\dot{Y} = 0$.

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The goal is to understand the evolution of the vector of eigenvalues of the configuration space variable $X_t = X + tY$, which can be regarded as a point in $\Delta_n^+ := \{(q_1, \dots, q_n) \in \mathbb{R}^n : q_1 \leq q_2 \leq \cdots \leq q_n\}$. The variation of the $q_t \in \Delta_n^+$ is clearly not dynamical, but one might hope to fit q in a higher-dimensional coordinate system which would be describable in dynamical terms.

In this regard, it is prudent to compute X and Y in a time-dependent orthonormal basis of $\mathbb{R}^n$ which consists of eigenvectors $e_1(t), \dots, e_n(t)$ of X_t, i.e., $X_t e_j(t) = q_j(t) e_j(t)$, where $q_t = \mathrm{diag}(q_1, \dots, q_n)$. Define p to be the diagonal of Y in the above basis, i.e., $p_j(t) = \langle e_j(t), Ye_j(t)\rangle$ and the remaining variables are the entries of Y weighted with the difference of the eigenvalues $l_{ij} = (q_j - q_i)\langle e_i, Ye_j\rangle$.

One directly verifies that these variables constitute a closed system, the first part of which is canonical. In fact (see [2]),

$$\dot{q}_j = p_j, \qquad \dot{p}_j = -2\sum_{k\neq j} \frac{l_{jk} l_{kj}}{(q_j - q_k)^3},$$
$$\dot{l}_{mj} = -\sum_{k \neq m, j} l_{mk} l_{kj} \left(\frac{1}{(q_m - q_k)^2} - \frac{1}{(q_j - q_k)^2} \right).$$

Thus the variables (q, p, l) give an extended system of desired type.

In fact, if $H := \|Y\|^2$ is transformed to these variables, i.e.,

$$\mathcal{H}(q, p, l) = \frac{1}{2}\sum_j p_j^2 + \frac{1}{2}\sum_{j\neq m} \frac{l_{mj}^2}{(q_m - q_j)^2},$$

then there is a simple Poisson bracket in which this system is defined by $\mathcal{H}$ as its Hamiltonian: the variables (q, p) are canonical, i.e., $\{p_i, p_j\} = \delta_{ij}$ and $\{p_i, p_j\} = \{q_i, q_j\} = 0$. They commute with l and

$$\{l_{\alpha\beta}, l_{ij}\} = \tfrac{1}{2}(\delta_{\alpha j} l_{\beta i} + \delta_{\beta i} l_{\alpha j} - \delta_{\beta j} l_{\alpha i} - \delta_{\alpha i} l_{\beta j}).$$

If one regards the variables (q, p, l) as being non-linear coordinates on $S := \Delta_n^+ \times Q$, and computes the Jacobian to change back to linear variables on S and then projects onto the eigenvalue variable q, one obtains the Gaußian orthogonal ensemble, i.e., the density $\prod_{i<j}(q_i - q_j)$. This is an indicator of the presence of random matrix theory (RMT).

1.2. Basic goals

The above computations are quite typical in physical settings where dynamical variables are essential ingredients. One of the goals of the present paper is to underline a more geometric side, the role of symmetry- and momentum-coordinates.

As explained in Section 3, it is at first appropriate to discuss this in terms of a naturally induced Poisson structure on the full quotient M/G of a symplectic manifold by a Lie group of symplectic diffeomorphisms. In the above example, M is just the cotangent bundle of Q, and G is $SO_n(\mathbb{R})$.

The "variable point of view" above amounts to choosing a special slice S for the quotient $M \to M/G$ which involves the coordinate of interest, i.e., q, as well as non-linear momentum-coordinates Ω. Another of our goals is to explain the nature of such a slice and show how it is constructed in a more general setting. It is interesting that in other cases a "thick slice" S is appropriate, i.e., one of higher dimension than that of M/G.

Computing in momentum-coordinates on naturally chosen submanifolds of S in Section 4, we give descriptions of the Poisson structures in the examples of symmetric, Hermitian and general complex matrices. The symmetric case was carried out in [2] as in Section 1.1. The latter cases were formally known, but not in mathematical detail (see [1,5]).

While the sets S are by no means invariant by the Hamiltonian flow, they do carry canonical measures. In the examples considered here there are image measures arising via orbital integration of standard invariant Gaußian measures on M (see Section 5). In all cases considered, when computed in non-linear momentum-coordinates, this is shown to be the uniform density on S and a change to linear coordinates and projection onto the eigenvalue variable yield densities which verify the presence of RMT. In fact, these calculations explain exactly why RMT is present.

2. Notation and basic structural information

In the sequel M denotes a connected manifold equipped with a Poisson structure $\{\,,\,\}: \mathcal{E}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$. Recall that $\{\,,\,\}$ is bilinear, alternating and that the Jacobi identity is satisfied, i.e., $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$. In order to regard $\{f, \cdot\} = V_f$ as a vector field on M one requires the validity of the Leibnitz-rule: $\{f, gh\} = h\{f, g\} + g\{f, h\}$.

The map $\mathcal{E}(M) \to \mathrm{Vect}(M)$, $f \mapsto V_f$, is easily shown to be local in the sense that V_f only depends on the differential $\mathrm{d}f$ and the Poisson structure is then defined by a linear map $P_{\{\,,\,\}}: T^*M \to TM$, $\mathrm{d}f(x) \mapsto V_f(x)$, which satisfies conditions which are equivalent to those listed above.

If $G \times M \to M$ is a smooth Lie group action, then $\{\,,\,\}$ is said to be invariant if $P_{\{\,,\,\}}$ is G-equivariant, i.e., $P_{\{\,,\,\}} \circ g = g \circ P_{\{\,,\,\}}$ or equivalently $\{g(f), g(h)\} = g(\{f, h\})$ for all $f, h \in \mathcal{E}(M)$ and all $g \in G$. Such invariant Poisson structures induce a Poisson structure on $\mathcal{E}(M)^G$, because the bracket of two invariant functions is again invariant. In case of confusion, we denote this by $\{\,,\,\}^G$.

Although it is not mandatory, it is of conceptual interest to associate a geometric model to $\mathcal{E}(M)^G$. The first candidate is the space of G-orbits $M/G := \{Gx : x \in M\}$ equipped with the quotient topology. If the G-action is proper, i.e., $G \times M \to M \times M$, $(g, x) \mapsto (g(x), x)$ is a proper map, then M/G is a Hausdorff differentiable space with mild singularities occurring in at most codimension 2. By definition $(\mathcal{E}(M)^G, \{\,,\,\}^G)$ is the Poisson structure on M/G.

A Hamiltonian system is a triple $(M, \{\,,\,\}, H)$ with $H \in \mathcal{E}(M)$. Given such a system, one is interested in qualitative aspects of the field V_H or of an associated discretized version. The first rough invariant is the subalgebra $Z(H) := \{f \in \mathcal{E}(M) : \{H, f\} = 0\}$ of constants of motion. The name is justified by the fact that $f \in Z(H)$ if and only if $V_H(f) = 0$, i.e., f is invariant under the local 1-parameter group action associated to V_H.

A Casimir function, which is by definition an element of the degeneracy $D_{\{\,,\,\}} := \{f \in \mathcal{E}(M) : \{H, f\} = 0\, \forall H \in \mathcal{E}(M)\}$, is a constant of motion for every system. Of course the constant functions $\mathbb{R}$ are always contained in $D_{\{\,,\,\}}$. If there are no other Casimirs, $D_{\{\,,\,\}} = \mathbb{R}$, then $\{\,,\,\}$ is said to be non-degenerate. Equivalently, $P_{\{\,,\,\}}$ defines a non-degenerate 2-form $\omega \in \mathcal{E}^2(M)$ which, due to the validity of the Jacobi identity, is closed. In other words, (M, ω) is a symplectic manifold with Hamiltonian fields being defined by $\mathrm{d}f = i_{V_f}\omega = \omega(V_f, \cdot)$. We adapt the sign convention $\{f, h\} := \omega(V_h, V_f)$.

The Poisson structure of a symplectic manifold is invariant with respect to a Lie group action $G \times M \to M$ if and only if $g^*\omega = \omega$ for all $g \in G$. If G is connected, then this can be expressed at the vector field level by $L_V\omega = 0$ for all fields V which are induced from the action. This formulation in terms of the Lie derivative allows one to apply Cartan's formula, $L_V = i_V \cdot d + d \cdot i_V$.

Thus, if ω is G-invariant and ξ_M is a field associated to the action of a 1-parameter subgroup, then $i_{\xi_M}\omega$ is a closed 1-form and at least locally there is an associated momentum function μ_ξ with $\mathrm{d}\mu_\xi = i_{\xi_M}\omega$. Note that if $H \in \mathcal{E}(M)^G$, then $\{H, \mu_\xi\} = V_H(\mu_\xi) = 0$, i.e., $\mu_\xi \in Z(H)$ is a constant of motion.

This formal version of Noether's principle leads one to bundle together all such μ_ξ into a single map $\mu : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra, Lie(G), equipped with the coadjoint representation.

Such a map is called a moment map if it is G-equivariant, and for every $\xi \in \mathfrak{g}$, the coordinate $\xi \circ \mu = \mu_\xi$ is a Hamiltonian of ξ_M as above. In general such a map may not exist, but in every example considered here its explicit definition will be obvious.

Since $\{H, \mu\} = 0$ for every invariant Hamiltonian H, it is suitable to use μ as a coordinate whenever possible. In this way the canonical Poisson structure $\{\,,\,\}_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ will appear in the Poisson structure, e.g., on M/G.

The structure $\{\,,\,\}_{\mathfrak{g}^*}$ is defined by the structure constants of the Lie algebra $\mathfrak{g}$. If $\xi, \eta \in \mathfrak{g}$ are regarded as linear functions, then $\{\xi, \eta\} := [\xi, \eta]$. By using Leibnitz's rule, this definition extends naturally to the symmetric algebra of polynomials and by density to the smooth functions $\mathcal{E}(\mathfrak{g}^*)$.

Equivalently, for $f, h \in \mathcal{E}(\mathfrak{g}^*)$ define $\{f, h\}_{\mathfrak{g}^*}(\alpha) := \alpha([\mathrm{d}f(\alpha), \mathrm{d}h(\alpha)])$, where $\mathrm{d}f(\alpha)$ and $\mathrm{d}g(\alpha)$ are regarded as elements of $\mathfrak{g}$. In the example of Section 1, the brackets $\{l_{mn}, l_{ij}\}$ represent the Poisson structure on (so_n^*).

All examples here involve cotangent structures on spaces of operators. In general, if Q is a manifold representing configuration space and $M = T^*Q$ is the associated phase space, then M comes equipped with its standard symplectic structure ω_{std}: for $v \in T_\alpha M$, where α is a cotangent vector at $q \in Q$, define $\theta \in \mathcal{E}^1(M)$ by $\theta(v) := \alpha(\pi_*(v))$. Here π denotes the canonical projection $\pi : M \to Q$, up to sign convention $\omega_{\mathrm{std}} = \mathrm{d}\theta$.

If $q = (q_1, \dots, q_n)$ is a coordinate system in Q and $p = (p_1, \dots, p_n)$ are the associated coordinates in the π-fibers, i.e., a 1-form is described by $p_1\,\mathrm{d}q_1 + \cdots + p_n\,\mathrm{d}q_n$, then $\omega_{\mathrm{std}} = \sum \mathrm{d}p_j \wedge \mathrm{d}q_j$.

In Sections 4 and 5, we carry out concrete calculations for cotangent bundles of vector spaces which with one exception are Lie algebras. For such a real vector space V, its cotangent bundle is $V \oplus V^*$ equipped with its standard structure. Since the individual factors $V \oplus \{0\}$ and $\{0\} \oplus V^*$ are Lagrangian, it is enough to describe the mixed terms: $\omega(v, w^*) = w^*(v)$. If G is a Lie group acting on V via a linear representation, then its action on $V \oplus V^*$ is symplectic.

The examples considered here correspond to selected elementary quantum mechanical models. The configuration space Q is, e.g., the space of symmetric, anti-Hermitian or complex matrices. Thus the eigenvalues are real, imaginary or mixed, with the latter case being regarded as a study of invariant dissipative systems (see [5]). The symplectic manifolds M in question are the associated cotangent vector spaces and the symmetry group is in each case an obvious compact group acting via its adjoint representation.

In all cases there exist invariant trace-pairings on the vector space in question. For $X, Y \in V = \Sigma_n, \mathfrak{su}_n, \mathfrak{gl}_n(\mathbb{C})$ the pairing $\langle X, Y\rangle := \mathrm{Re}(\mathrm{tr}(XY)^\dagger)$, where $Y^\dagger := \bar{Y}^{\mathrm{T}}$ is non-degenerate. Thus the phase spaces M can be taken to be $\Sigma_n \times \Sigma_n$, $\mathfrak{su}_n \times \mathfrak{su}_n$ and $\mathfrak{gl}_n \times \mathfrak{gl}_n$ with the respective transported structures $\omega = \mathrm{Re}(\mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X^\dagger))$. Here we use the obvious conventions for matrix-valued differential forms.

The G-action is then just the diagonal action by conjugation on pairs (X, Y) of matrices. In the case of Σ_n, the group G is SO_n and otherwise G is the unitary group SU_n.

In the former cases, the moment map is given by $\mu(X, Y) = [Y, X]$ and in the case of complex matrices by $\frac{1}{2}([Y, X^\dagger] + [Y^\dagger, X])$. Of course $\mathfrak{su}_n \times \mathfrak{su}_n$ is a symplectic subspace of $\mathfrak{gl}_n \times \mathfrak{gl}_n$. So it is possible to derive results for the former by restricting from the latter.

It would seem likely that all of our considerations can be carried out for any compact real algebra, for other representations and in certain non-linear contexts. It would be of particular interest to discuss these matters in the indefinite case, i.e., for non-compact real forms.

3. Symplectic reduction and the Poisson structure on M/G

We now turn to a description of the Poisson structure on $N = M/G$. Since we are willing to restrict to the generic points where the orbit-dimension is constant, this can be given in terms of a symplectic foliation.

In general, the rank of a Poisson structure $\{\,,\,\}$ at a point x in a Poisson manifold N is defined to be the rank of the induced linear map $P_{\{\,,\,\}} : T_x^*N \to T_xN$.

Lemma 3.1. *If* $\{\,,\,\}$ *is of constant rank, then* $\mathrm{Im}(P_{\{\,,\,\}})$ *is an integrable subbundle of TN.*

Proof. The image $\mathrm{Im}(P_{\{\,,\,\}})$ consists of tangent vectors of the form $V_f(x)$ with $f \in \mathcal{E}(M)$. Since $\{f, h\}$ yields the field $[V_f, V_h]$, the integrability is immediate. □

If $\mathcal{L}$ is a leaf of the induced foliation $\mathcal{F}$ of a constant rank Poisson structure, then $\{\,,\,\}_{\mathcal{L}}$ is well defined by extending functions $f, h \in \mathcal{E}(\mathcal{L})$, applying $\{\,,\,\}$ and restricting. Of course this is carried out at the germ level. Since $\{\,,\,\}_{\mathcal{L}}$ is non-degenerate, $\mathcal{F}$ is called the associated symplectic foliation.

It is convenient to describe the symplectic foliation of M/G via the symplectic reduction of M. Let us recall this procedure.

Let (M,ω) be a symplectic manifold with a smooth action $G \times M \to M$ of a connected Lie group of symplectic diffeomorphisms. Assume that there exists a moment map $\mu : M \to \mathfrak{g}^*$. For $x \in M$, let $\alpha := \mu(x)$ and F_α be the fiber $\mu^{-1}\{\alpha\}$. Let G_x and G_α denote the respective isotropy groups.

Direct application of the definitions yields the basic formula

$$\mathrm{d}\mu(x)(v)(\xi) = \omega(x)(\xi_M, v)$$

for $v \in T_x M$, $\xi \in \mathfrak{g}$ and ξ_M the associated field. In particular, $\mathrm{Ker}(\mathrm{d}\mu(x)) = (T_x Gx)^{\perp\omega}$ and the notion of locally constant rank corresponds to locally constant orbit dimension.

Of course F_α may be singular, but here we always restrict to a possibly smaller (dense, open) subset of M where orbit-dimension is locally constant, e.g., where μ is of constant rank and F_α is smooth. The form $\omega_\alpha := i^*_{F_\alpha}\omega$ will usually have a certain degree of degeneracy.

Lemma 3.2. *The degeneracy* $(T_x F_\alpha)^{\perp\omega_\alpha}$ *is* $T_x G_\alpha x$.

Proof. As was noted above, $(T_x F_\alpha)^{\perp\omega} = T_x Gx$. Thus,

$$(T_x F_\alpha)^{\perp\omega_\alpha} = T_x F_\alpha \cap T_x Gx = T_x G_\alpha x. \qquad \square$$

In general, if ω is a closed 2-form on a manifold N with $TN^{\perp\omega}$ a constant rank bundle, then it is integrable: for X, Y vector fields from this degeneracy and $Z \in \mathrm{Vect}(N)$ arbitrary, a direct calculation of $0 = \mathrm{d}\omega(X, Y, Z)$ shows that $\omega([X, Y], Z) = 0$.

Thus $TN^{\perp\omega}$ defines a foliation $\mathcal{F}$. Furthermore, if the quotient $N \to N/\mathcal{F}$ exists as a differentiable manifold, then ω pushes down to a symplectic form. For this it is enough to show that $L_X\omega = 0$ for all X in the degeneracy, but this is immediate from Cartan's formula.

In the above case, the degeneracy foliation is defined by the orbits in F_α of the connected component G°_α of the momentum isotropy. Since we always restrict to sets of points where $\pi : M \to M/G$ exists as a differentiable manifold, we may assume that these orbits are closed: $F_\alpha/G^\circ_\alpha =: \tilde{N}_\alpha$ *is the symplectic reduction of the* μ*-fiber* F_α.

If F_{α_1} and F_{α_2} are μ-fibers over the same coadjoint orbit B, then there exists $g \in G$ with $g(F_{\alpha_1}) = F_{\alpha_2}$. This map is compatible with symplectic reduction. Furthermore, if $L := \mu^{-1}(B)$ is the G-invariant momentum level over B, then $\pi(L) = \pi(F_{\alpha_1}) = \pi(F_{\alpha_2})$. Up to a quotient by a discrete group, this is just the reduced space $\tilde{N}_\alpha : \pi(L) =: N_\alpha = \tilde{N}_\alpha/\Gamma$, where $\Gamma = G_\alpha/G^\circ_\alpha$. In fact these manifolds define the symplectic foliation on $N := M/G$ of the quotient Poisson structure.

Proposition 3.3. *Assume that the* G*-action has locally constant orbit-dimension, that the orbits are closed and that* $\pi : M \to M/G =: N$ *is a constant rank map onto a smooth manifold so that* $\mathcal{E}(N) = \mathcal{E}(M)^G$. *Suppose that the coadjoint orbits in the image* $\mu(M)$ *are parameterized by a set* $\mathcal{P}$. *Then* $N = \cup_{\alpha\in\mathcal{P}} N_\alpha$ *is the symplectic foliation of* N.

Proof. Since disjoint G-invariant subsets of M are mapped by π onto disjoint subsets of N, for $\alpha_1 \neq \alpha_2$ different parameter values in $\mathcal{P}$, it follows that N_{α_1} and N_{α_2} are disjoint submanifolds of N.

Now, for $x \in M$ with $y := \pi(x)$, the image $\mathrm{Im}(P_{\{,\}}) \subset T_y N$ is defined by $\pi_*(\{V_f(x) : f \in \mathcal{E}(M)^G\})$. Recall that $\{f, \mu\} = 0$ for $f \in \mathcal{E}(M)^G$, i.e., $V_f(x) \in T_x F_\alpha$ for $f \in \mathcal{E}(M)^G$.

Furthermore, for $k := \operatorname{codim} Gx = \dim F_\alpha$, there exist $f_1, \dots, f_k \in \mathcal{E}(M)^G$ with $(\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_k)(x) \neq 0$. In other words, the fields $V_{f_1}, \dots, V_{f_k}$ form a basis of $T_x F_\alpha$. Consequently, at $y \in N$ the image of $P_{\{,\}}$ is $T_y \pi(F_\alpha) = T_y N_\alpha$. □

4. Slice coordinates and computation of Poisson structures

For invariant Hamiltonian systems, the completely reduced phase space is $N = M/G$ equipped with the Poisson structure defined by the algebra $\mathcal{E}(M)^G$ of invariant functions. Under genericity assumptions, Proposition 3.3 gives an abstract description of the symplectic foliation of this structure. For applications, it is important to determine coordinates in which the Poisson structure can be explicitly computed.

For these purposes a (possibly thick) slice $S \subset M$ is defined to be a submanifold with the property that S is transversal to $\pi : M \to M/G$ at its generic points, i.e., in the sense of Proposition 3.3, and that $GS = M$. In all examples considered here, the *thickness* is controlled by a subgroup $T < G$: generically, $Gs \cap S = Ts$. If the action map defines a diffeomorphism $G \times S \cong M$, then we refer to S as an exact slice.

From now on we will only consider selected examples. In this section, we carry out calculations of the Poisson structures and in the following one we compute slice-density functions.

4.1. Symmetric operators

In the case of the orthogonal group $G = SO_n(\mathbb{R})$ acting diagonally by conjugation on $M = \Sigma_n \times \Sigma_n$, where the cotangent bundle structure on M is defined by the identification $\Sigma_n \cong \Sigma_n^*$ of the second factor via the pairing $\langle X, Y\rangle = \mathrm{tr}(XY)$, we choose $S := \Delta_n \times \Sigma_n$ with Δ_n being the set of diagonal matrices in the first factor.

Restricting our attention to generic points, we assume that no two eigenvalues of X are equal. Thus we shrink S to $\Delta_n^+ \times \Sigma_n$, where $q = \mathrm{diag}(q_1, \dots, q_n) \in \Delta_n^+$ satisfies $q_1 < \cdots < q_n$. Thus S is an exact slice and M is identified with $G \times S$ by the action map $(g, s) \to g(s)$.

Since the eigenvalue space is a factor of S, the first goal is reached. The extended coordinates are defined as follows: for $(D, Y) \in S$, let $l := l(D, Y)$ be the coordinates of the moment map $\mu(D, Y) = [Y, D]$ and define $p = p(D, Y) = \mathrm{diag}(p_1, \dots, p_n)$ to be the diagonal coordinates of Y.

Extend the coordinates q, p and l to be invariant functions on M, e.g., $q(g(s)) := q(s)$ for all $g \in G$. Regard $g \in G$ as an orthogonal matrix O and let (q, p, l, O) be a global coordinate system with values in various sets of matrices.

If W is defined to be the differential $O^{-1}\,\mathrm{d}O$, then a straightforward calculation shows that

$$\omega = \mathrm{tr}(\mathrm{d}p \wedge \mathrm{d}q) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W).$$

We carry out the completely analogous calculation in the case of the unitary group below. Hence, we omit it here.

The functions (q, p, l) are optimal coordinates on the complete reduction M/G. The Poisson structure there splits with the pair (q, p) being canonical, i.e., the standard structure on $\mathbb{R}^{2n}$, commuting with l and l itself having the coadjoint Poisson structure of $\mathfrak{so}_n$.

The latter point requires some discussion. In general, the moment map $\mu : M \to \mathfrak{g}^*$ is a Poisson map in the sense that for $f, h \in \mathcal{E}(\mathfrak{g}^*)_-\, \{\mu^*(f), \mu^*(h)\}_M = \mu^*(\{f, h\}_{\mathfrak{g}^*})$. Here this is also true for the invariant map l. This does not follow immediately from the analogous property for μ, but is closely related. In any case, this shows that the commutation relations for l are just given by pulling back the standard basis of linear functions on $\mathfrak{g}^*$ as was described in Section 1. Again the calculations are exactly the same as those in the unitary cases below. Thus, we omit them.

4.2. *Anti-Hermitian and general complex matrices*

We shall identify the Lie algebra $\mathfrak{su}_n$ with the space of trace-free, anti-Hermitian matrices. Since it is convenient to discuss this case at the same time as that for complex matrices, in the latter situation we always restrict to the trace-free case, i.e., $M := \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$. Since $\mathfrak{su}_n \times \mathfrak{su}_n$ is an SU_n-invariant symplectic subspace of M, all results will follow by restriction. Thus we begin with M as above and the diagonal action of $G = SU_n$ by conjugation.

Let $\mathfrak{b}_n$ be the subalgebra of upper-triangular matrices in $\mathfrak{sl}_n(\mathbb{C})$ and provisionally define $S := \mathfrak{b}_n \times \mathfrak{sl}_n$. Of course, for every element $X \in \mathfrak{sl}_n$ there exists $g \in G = SU_n$ such that $g(X) \in \mathfrak{b}_n$. However, g is not unique, i.e., this is the case of a thick slice.

Let T be the subgroup of diagonal matrices in G. It follows that S is T-invariant, and if $s \in S$ is generic and $g(s) \in S$, then $g \in T$.

For the purposes of computing the Poisson structure on M/G it is appropriate to choose an exact slice. For this and later discussions, let E_{ij} be the matrix (δ_{ij}^{mn}) and $\mathfrak{g}_i$ be the one-dimensional complex subspace generated by $E_{i(i+1)}$. Finally, let $R := \oplus \mathfrak{g}_i$.

Since the G-action is defined by conjugation, the (finite) center of G acts trivially and we may replace it by PSU_n. We do this (without changing the notation) and therefore may assume that T acts faithfully on R.

Now we replace S by a shrunken version to obtain an exact slice: in the second factor of $\mathfrak{b}_n \times \mathfrak{sl}_n$ we replace R by $R^+ := \{v = (v_1, \dots, v_{n-1}) \in R : v_i > 0\, \forall i\}$. Since $TR^+ = R$, it still follows that $GS = M$. Furthermore, for $s \in S$ generic, $Gs \cap S = \{s\}$.

Note also that $S_{\mathfrak{su}_n} := S \cap (\mathfrak{su}_n \times \mathfrak{su}_n)$ is an exact slice in $\mathfrak{su}_n \times \mathfrak{su}_n$. Concretely, $S_{\mathfrak{su}_n} = q \times \mathfrak{su}_n^+$, where q is the set of (trace-free, imaginary) diagonal matrices and $A = (A_{ij}) \in \mathfrak{su}_n^+$ whenever $A_{i(i+1)} > 0$ (resp. $A_{(i+1)i} < 0$), $i = 1, \dots, n-1$.

Returning to the case of $M = \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$, we choose functions on S as follows: $Z = (Z_{ij})_{i\le j}$ (resp. $P = (P_{ij})_{i\le j}$) are the standard complex linear coordinates of the upper-triangular parts of the first (resp. second) factors. Of course $P_{i(i+1)} > 0$ for $i = 1, \dots, n-1$. Let $l := \mu|S$ be the restriction of the moment map.

Now extend these to G-invariant functions on M by the canonical procedure, i.e., for $f \in \mathcal{E}(S)$, extend it to M by $f(g(s)) := f(s)$.

Finally, regard $g \in G$ as a unitary matrix U. For $x = g(s)$ define $U(x)$ to be the matrix of g.

Let us now compute the standard symplectic form $\omega = \mathrm{Re}(\mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X^\dagger))$ using these functions. For this it is convenient to regard $\omega = \mathrm{d}\theta$, where $\theta := \mathrm{Re}(\mathrm{tr}(Y\,\mathrm{d}X^\dagger))$. For temporary use let V denote the standard matrix coordinate of the second factor of S regarded in the usual way as an invariant function on M. Define $\mathrm{d}U$ to be the matrix of 1-forms $\mathrm{d}U_{ij}$ and $W := U^\dagger\,\mathrm{d}U$. Note that $W = -W^\dagger$ and that $\mathrm{d}W = -W \wedge W$.

The conversion of ω is a simple matter:

$$\begin{aligned}\theta &= \mathrm{Re}(\mathrm{tr}(UVU^\dagger\,\mathrm{d}(UZ^\dagger U^\dagger))) = \mathrm{Re}(\mathrm{tr}(V(\mathrm{d}Z^\dagger + [Z^\dagger, W^\dagger])))\\ &= \mathrm{Re}(\mathrm{tr}(V\,\mathrm{d}Z^\dagger)) - \mathrm{tr}(lW).\end{aligned}$$

Therefore,

$$\omega = \mathrm{Re}(\mathrm{tr}(\mathrm{d}V \wedge \mathrm{d}Z^\dagger)) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W).$$

Due to the upper-triangular nature of Z, the lower-triangular part of $\mathrm{d}V$ is not involved. Thus the first term in the description of ω can be replaced by $\mathrm{Re}(\mathrm{tr}(\mathrm{d}P \wedge \mathrm{d}Z^\dagger))$ and can be expanded as

$$\frac{1}{2}\sum_{i\le j}(\mathrm{d}P_{ij} \wedge \mathrm{d}\bar{Z}_{ij} + \mathrm{d}\bar{P}_{ij} \wedge \mathrm{d}Z_{ij}).$$

Since $P_{i(i+1)}$ is real-valued, there are terms of the form $\mathrm{d}P_{i(i+1)} \wedge \mathrm{d}\,\mathrm{Re}(Z_{i(i+1)})$. Thus we modify Z by replacing $Z_{i(i+1)}$ by $\mathrm{Re}(Z_{i(i+1)})$, $i = 1, \dots, (n-1)$.

It also follows that (Z, P, l, U) is now a global coordinatization of an appropriately defined dense open set in M. Using the obvious convention, we write

$$\omega = \mathrm{Re}(\mathrm{tr}(\mathrm{d}P \wedge \mathrm{d}Z^\dagger)) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W).$$

By restriction, we have global coordinates (q, p, l, U) on $\mathfrak{su}_n \times \mathfrak{su}_n$, where q and p are diagonal matrices in the first and second factors of $S_{\mathfrak{su}_n}$ and l is the restriction of the moment map. Of course these are extended to $M_{\mathfrak{su}_n}$ as invariant functions. Here it should be noted that l only takes on values in $\mathfrak{t}^\perp = \{A \in \mathfrak{su}_n : \mathrm{diag}(A) = 0\}$. Furthermore, we have the relations $\mathrm{Im}(q_i^{-1} l_{i(i+1)}) = 0$.

We regard (Z, P, l) as coordinates on the completely reduced space M/G. Due to the particularly simple form of ω, it is an easy job to describe the Poisson structure in these variables. The results can be summarized as follows.

Proposition 4.1. *The map $l : M \to \mathfrak{g}^*$ is a Poisson morphism, i.e., for f and h functions of l alone,*

$$\{f, h\}(l) = \mathrm{tr}\left(l\left[\left(\frac{\partial f}{\partial l}\right), \left(\frac{\partial h}{\partial l}\right)\right]^{\mathrm{T}}\right).$$

Remark. Although we have discussed this result in the context of complex matrices, it follows by restriction for $\mathfrak{su}_n \times \mathfrak{su}_n$ and via an analogous proof for $\Sigma_n \times \Sigma_n$.

Before carrying out the calculation, let us clarify the matrix notation. As usual, regard l as a complex matrix valued map with values in $\mathfrak{su}_n$, i.e., $l^\dagger = -l$, and let $\mathrm{d}l = (\mathrm{d}l_{ij})$ be the matrix of $\mathbb{C}$-valued 1-forms.

With the slice restrictions on Z and P, the entries of the matrices $\mathrm{d}Z, \mathrm{d}\bar{Z}, \mathrm{d}P, \mathrm{d}\bar{P}, \mathrm{d}l$ and W form a basis of the complex valued 1-forms at each point of M. We consider the dual basis of fields, e.g., $(\partial/\partial l)$ which satisfies $(\partial/\partial l_{ji}) = -(\partial/\partial \bar{l}_{ij})$. By abuse of notation, let $(\partial/\partial W)$ denote the fields dual to $W : W_{ij}(\partial/\partial W_{kl}) = \delta^{ij}_{kl}$. Of course $(\partial/\partial W)^\dagger = -(\partial/\partial W)$.

For the calculation of the Poisson brackets of functions of l alone, it is only necessary to consider the pieces of fields which involve $(\partial/\partial l)$ and $(\partial/\partial W)$.

For a real-valued field Z, it follows that

$$Z = \cdots + \mathrm{tr}\left(Z^l\left(\frac{\partial}{\partial l}\right)^{\mathrm{T}}\right) + \mathrm{tr}\left(Z^W\left(\frac{\partial}{\partial W}\right)^{\mathrm{T}}\right),$$

where Z^l and Z^W are also matrices in $\mathfrak{su}_n$.

We now compute the Hamiltonian field V_f of a real-valued function $f = f(l)$. For Z any $\mathbb{R}$-field,

$$\omega(V_f, Z) = -\mathrm{tr}(\mathrm{d}l \wedge W - lW \wedge W)(A, B),$$

where $A = \mathrm{tr}(V_f^l(\partial/\partial l)^{\mathrm{T}}) + \mathrm{tr}(V_f^W(\partial/\partial W)^{\mathrm{T}})$ and $B = \mathrm{tr}(Z^l(\partial/\partial l)^{\mathrm{T}}) + \mathrm{tr}(Z^W(\partial/\partial W)^{\mathrm{T}})$. Direct calculation yields $\omega(V_f, Z) = \mathrm{tr}(CZ^W) + \mathrm{tr}(DZ^l)$, where $C = -V_f^l + [l, V_f^W]$ and $D = V_f^W$.

Now $\mathrm{d}f(Z) = \mathrm{tr}(Z^l(\partial f/\partial l)^{\mathrm{T}})$, and therefore the Hamiltonian condition $\mathrm{d}f(Z) = \omega(V_f, Z)$ yields $V_f^W = (\partial f/\partial l)^{\mathrm{T}}$ and $C = 0$, i.e., $V_f^l = [l, (\partial f/\partial l)^{\mathrm{T}}]$.

In summary

$$V_f = \mathrm{tr}\left(\left[l, \left(\frac{\partial f}{\partial l}\right)^{\mathrm{T}}\right]\left(\frac{\partial}{\partial l}\right)^{\mathrm{T}} + \left(\frac{\partial f}{\partial l}\right)^{\mathrm{T}}\left(\frac{\partial}{\partial W}\right)^{\mathrm{T}}\right).$$

The formula for $\omega(V_f, V_h)$ in Proposition 4.1 follows by direct evaluation.

5. Slice densities

Our goal here is to compute the measure on S (resp. M/G) which is naturally related to the Liouville measure $\mathrm{d}\lambda_M$ associated to the volume form $\omega_M = \omega^n$ on M.

For S an exact or thick slice and ω_S its linear volume form, the *slice density* $\rho : S \to \mathbb{R}^{\geq 0}$ is uniquely defined by the G-invariant functions: for every compactly supported function $f \in \mathcal{E}_0(M)^G$,

$$\int_M f\omega_M = \int_S f\rho\omega_S.$$

If, e.g., $p\,\mathrm{d}\lambda_M$ is a G-invariant probability distribution on M, e.g., $\exp(-\|\cdot\|^2)\,\mathrm{d}\lambda_M$, then the appropriate density on S is $p\rho\,\mathrm{d}\lambda_S$. In this way, it is a simple matter to compute the desired image measure via the linear projection $S \to \mathfrak{t}^+$ to the eigenvalues.

In order to give a unified description of our calculations it is convenient to introduce some notation. In the case of $M = \Sigma_n \times \Sigma_n$, where $S = \Delta_n^+ \times \Sigma_n$, let (q, p, l) be the coordinates defined above.

In the cases of complex matrices $M = \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$ and the restriction to $M_{\mathfrak{su}_n} = \mathfrak{su}_n \times \mathfrak{su}_n$, it is prudent to use thick slices. For complex matrices, we choose $S := \mathfrak{b}_n^+ \times \mathfrak{sl}_n(\mathbb{C})$ and $S_{\mathfrak{su}_n} = \mathrm{i}\Delta_n^+ \times \mathfrak{su}_n$ its intersection with $M_{\mathfrak{su}_n}$.

In the former case, for $(X, Y) \in S$, let q denote the linear coordinates of X, p the linear coordinates of the upper-triangular part of Y, and l the non-diagonal coordinates of the moment map. In the latter case, q represents the linear coordinates of the first factor, p the diagonal of second and l the moment map. These are in fact just the restricted coordinates from the space of complex matrices.

Proposition 5.1. *In the coordinates* (q, p, l) *the canonical slice measure is given by* $\rho\,\mathrm{d}\lambda_S = \mathrm{d}q\,\mathrm{d}p\,\mathrm{d}l$.

This interesting fact, i.e., that in non-linear momentum-coordinates the canonical measure is Euclidean, is not proved directly. Instead, we directly compute ρ. This is of course what is needed for the calculation of eigenvalue densities. For example, for H the invariant norm function and given density $\exp(-\frac{1}{2}H)\,\mathrm{d}\lambda_M$, we obtain, respectively, the Gaußian orthogonal, Gaußian unitary and Ginibre ensembles.

5.1. Orthogonal ensembles

In this case S is an exact slice, i.e., the action map $\alpha : G \times S \to M$, $(g, s) \mapsto g(s)$, is a diffeomorphism. Let ω_G be the normalized bi-invariant volume form on the orthogonal group $G = SO_n$ with $\int_G \omega_G = 1$ and ω_S be the standard Euclidean volume form on S defined by the linear coordinates.

The Jacobian ρ is defined on $G \times S$ by $\rho\omega_G \wedge \omega_S = \alpha^*\omega_M$. Since all of the differential forms which are involved are G-invariant, it follows that ρ is likewise G-invariant, i.e., $\rho = \rho(s)$ is defined on the slice S. Furthermore, for $f \in \mathcal{E}_0(M)^G$, it follows from Fubini's Theorem that

$$\int_M f\omega_M = \int_{G\times S} \rho\alpha^*(f)\omega_G \wedge \omega_S = \int_S \rho f\omega_S.$$

Thus ρ is the desired density.

We compute ρ along $\{e\} \times S$ in $G \times S$. For this let $\mathcal{F}_0$ be the usual orthonormal frame for $T_eG = \mathfrak{so}_n = \{A : A + A^{\mathrm{T}} = 0\}$, i.e., $\mathcal{F}_0 = \langle J_{ij} : i < j\rangle$, where $J_{ij} = E_{ji} - E_{ij}$. Recall that $E_{ij} = (\delta_{ij}^{mn})$ and note that we do not have explicitly normalized lengths.

The linear splitting $\Sigma_n \times \Sigma_n = \mathfrak{so}_n \times S$ is convenient for computing ρ. It is enough to compute the image frame $\alpha_*(\mathcal{F}_0)$ at a point $(0, s)$, and then ρ is just the determinant of its projection $\mathrm{Pr}(\alpha_*(\mathcal{F}_0))$ on the first factor $\mathfrak{so}_n$.

Since the action is defined by conjugation of matrices, it follows that $\mathrm{Pr}(\alpha_*(J_{ij})) = [J_{ij}, q]$, where $q = (q_1, \ldots, q_n)$ is the diagonal in the first factor of S. Thus $\mathrm{Pr}(\alpha_*(J_{ij})) = (q_j - q_i)J_{ij}$, and it follows that $\rho(s) = \prod_{i<j}(q_j - q_i)$. Using the standard norm function H on M, this yields as an image measure on $\mathfrak{t}^+$ the Gaußian orthogonal ensemble.

Now the moment map on S is given by $l(q, Y) = [Y, q]$. Therefore, it follows immediately that $\rho(s)\,\mathrm{d}\lambda_S = \mathrm{d}q\,\mathrm{d}p\,\mathrm{d}l$ as stated in Proposition 5.1.

5.2. *Fiber integration*

The computation of slice densities in the remaining cases is technically slightly different from the above, because the map $\alpha : G \times S \to M$ is no longer a diffeomorphism but rather a T-principal bundle.

In this situation $\alpha : G \times S \to M$ is defined as the quotient by the diagonal T-action $t(g, s) \mapsto (gt^{-1}, t(s))$. Let $\mathcal{T}$ be the associated invariant frame field along the fibers.

As in the case of an exact slice, let ω_G be the normalized invariant volume form on G and ω_S the standard Euclidean volume form on S. Define the function ρ by the identity

$$\rho i_{\mathcal{T}}(\omega_G \wedge \omega_S) = \alpha^*\omega_M.$$

Here $i_{\mathcal{T}}$ denotes contraction with the frame $\mathcal{T}$. For the same reason as above, it follows that $\rho = \rho(s)$ is G-invariant.

Applying fiber integration for $f \in \mathcal{E}_0^G$,

$$\int_{G\times S} \alpha^*(f)\rho\omega_G \wedge \omega_S = \int_M f\left(\int_T t^*(\rho i_{\mathcal{T}}(\omega_G \wedge \omega_S)\,\mathrm{d}t\right) = \int_M f\omega_M,$$

and applying Fubini's Theorem as in the previous case,

$$\int_{G\times S} \alpha^*(f)\rho\omega_G \wedge \omega_S = \int_S f\rho\omega_S.$$

Thus ρ is the desired slice density.

5.3. *Unitary ensembles*

The computation of the slice density ρ in the case of the thick slice $S = \mathrm{i}\Delta_n^+ \times \mathfrak{su}_n$ in the unitary case goes essentially the same as that for the orthogonal group. Here, however, we must replace the full orthogonal frame on G by a frame $\mathcal{F}^\perp$ for $\mathfrak{t}^\perp$. For this let $J_{ij} = E_{ji} - E_{ij}$, $K_{ij} = \sqrt{-1}(E_{ji} + E_{ij})$ and $\mathcal{F}^\perp := \langle J_{ij}, K_{ij}; i < j\rangle$.

At $s \in S$, we have the induced frame $\alpha_*(\mathcal{F}^\perp)$ defined by the action. Using the local splitting $M = \mathfrak{t}^\perp \times S$ along S, we must only compute the determinant of the projection $\mathrm{Pr}(\alpha_*(\mathcal{F}^\perp))$ onto $\mathfrak{t}^\perp$.

Since $\mathrm{Pr}(\alpha_*(J_{ij})) = [J_{ij}, q] = (q_j - q_i)K_{ij}$ and $\mathrm{Pr}(\alpha_*(K_{ij})) = [K_{ij}, q] = (q_j - q_i)J_{ij}$, where $q = \mathrm{diag}(\mathrm{i}q_1, \ldots, \mathrm{i}q_n)$, it immediately follows that $\rho(s) = \prod_{i<j}(q_j - q_i)^2$.

This yields the Gaußian unitary ensemble on $\mathfrak{t}^+$. Furthermore, since $\mu(q, Y) = [Y, q]$, changing to momentum-coordinates, it again follows that $\rho(s)\, d\lambda_S = \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}l$ as required by Proposition 5.1.

5.4. Complex matrices

Conceptually speaking, the case $M = \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$ of complex matrices is handled in exactly the same way as the others. The only technical difference is that the projected frame is in the triangular form.

For the computations, let $\mathfrak{n}$ be the subalgebra of strictly lower-triangular complex matrices. The thick slice is $\mathfrak{b}^+ \times \mathfrak{sl}_n$. Thus, we again have the natural projection onto $\mathfrak{n}$ along S, and we must compute the determinant of $\mathrm{Pr}(\alpha_*\mathcal{F}^\perp))$, where $\mathcal{F}^\perp$ is defined as above.

For this it is convenient to let

$$\langle E_{21}, E_{32}, \ldots, E_{(n-1)n}; E_{31}, \ldots, E_{n(n-2)}; \ldots; E_{n1}\rangle$$

be the ordered basis for $\mathfrak{n}$.

Let $\mathfrak{n}_1 := [\mathfrak{n}, \mathfrak{n}]$, $\mathfrak{n}_2 = [\mathfrak{n}_1, \mathfrak{n}]$, etc., and define the complements $\mathfrak{c}_k$, $k = 1, \ldots, n-1$, by $\mathfrak{n}_{(k-1)} = \mathfrak{n}_k \oplus \mathfrak{c}_k$. Of course we use $\langle E_{k1}, \ldots, E_{n(n-k+1)}\rangle$ as a basis for $\mathfrak{c}_k$.

Let $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $Z = (Z_{ij})_{i<j}$ be the strictly upper-triangular matrix with coordinates Z_{ij}.

Direct computation shows that

$$\mathrm{Pr}(\alpha_*(J_{ij})) = \mathrm{Pr}([J_{ij}, \lambda + Z]) = (\lambda_j - \lambda_i)E_{ji} + \text{terms in } n_{(j-i+1)},$$

and $\mathrm{Pr}(\alpha_*(K_{ij})) = \sqrt{-1}\,\mathrm{Pr}(\alpha_*(J_{ij}))$. Thus, the projected frame in $\mathfrak{t}^\perp$ is $(\mathcal{F}, \mathrm{i}\mathcal{F})$ where $\mathcal{F}$ is in upper-triangular form in the above basis.

Since the diagonal entries of $\mathcal{F}$ consist of all possible differences $\lambda_j - \lambda_i$, it follows that the determinant of $(\mathcal{F}, \mathrm{i}\mathcal{F})$ over the real numbers is $\prod_{i<j}|\lambda_j - \lambda_i|^2 = \rho(s)$.

In this case, the calculation of the Jacobian of the change of variables to momentum-coordinates requires a bit of care but is nevertheless computed in a straightforward way (see [5]). Again this yields $\rho\, \mathrm{d}\lambda_S = \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}l$ as claimed in Proposition 5.1. Analogous to the orthogonal and unitary cases, this slice density leads to Ginibre's distribution of eigenvalues.

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